Skill Management in Large-scale Service Marketplaces

Online Supplement

Appendix S.1: Proofs in Section 5

S.1.1. Proof of Proposition 1

We prove our claim by considering two cases: i)$\rho_i \le \overline{\alpha}_i$ and ii) $\rho_i > \overline{\alpha}_i$.

i) $\rho_{\mathbf{i}} \le \overline{\alpha}_{\mathbf{i}}$: In this case, for any $D_i \le 1$, we have that waiting times are zero, so that $U_i^f(\mathbf{r_i}, \mathbf{y_i}, \mathbf{t_i}, D_i) \ge r_{i_{N_i}} \ge \underline{u}$. Thus, we should have that $D_i^{ce} = 1$.

ii) $\rho_{\mathbf{i}} > \overline{\alpha}_{\mathbf{i}}$: In this case, we have that $U_i^f(\mathbf{r_i}, \mathbf{y_i}, \mathbf{t_i}, D_i) \ge r_{i_{N_i}} \ge \underline{u}$ for any $D_i \le \overline{\alpha}_i/\rho_i$, similar to part i. On the other hand, for any $D_i > \overline{\alpha}_i/\rho_i$, we have that waiting times are infinity, so that $U_i^f(\mathbf{r_i}, \mathbf{y_i}, \mathbf{t_i}, D_i) = -\infty < \underline{u}$, since $\rho_i > \overline{\alpha}_i$. Hence, we should have that $D_i^{ce} = \overline{\alpha}_i/\rho_i$.

Agent utilizations: When $D_i^{ce} = 1$, we have that $\sigma_{i_\ell}^{ce}(\mathbf{r_i}, \mathbf{y_i}, \mathbf{t_i}) = \sigma_\ell^f(\mathbf{r_i}, \mathbf{t_i} \circ \mathbf{y_i}/\overline{\alpha}_i, \rho_i/\overline{\alpha}_i)$. Moreover, $\rho_i D_i^{ce} = \overline{\alpha}$ when $D_i^{ce} < 1$, so that $\sigma_{i_\ell}^{ce}(\mathbf{r_i}, \mathbf{y_i}, \mathbf{t_i}) = 1$ for all $\ell \in \{1, \ldots, N_i\}$. We obtain the function presented in the proposition when we combine these two observations.

S.1.2. Proof of Theorem 1

In this proof, we, first, focus on symmetric *Market Equilibrium*, where the same type of agents (high-value or low-value) charge the same price. We let $(p_H, p_L; \alpha_H, \alpha_L)$ be a symmetric *Market Equilibrium* under the special market structure. We will provide a discussion on non- symmetric *Market Equilibrium* at the end of the proof. As a first step towards characterizing the symmetric equilibrium, we derive the revenue of agents when all the agents in the same pool charge the same price as follows:

Corollary 2. *Let $V_H^{ce}(p_H, p_L)$ and $V_L^{ce}(p_H, p_L)$ be the revenue of a high-value and a low-value agent, respectively, when all the high-value agents charge p_H and all low-value agents charge p_L. If $R_H - p_H \ne R_L - p_L$, then we have that*

$$V_h^{ce}(p_H, p_L) = \begin{cases} p_H(\rho_s/\alpha_H)\tau_s & \text{if } \rho_s \le \alpha_H \\ p_H\tau_s & \text{if } \rho_s > \alpha_H \end{cases}, \text{ and } V_l^{ce}(p_H, p_L) = \begin{cases} 0 & \rho_s \le \alpha_H \\ p_L(\rho_s - \alpha_H)/\alpha_L\tau_s & \alpha_H < \rho_s < \alpha_H + \alpha_L \\ p_L\tau_s & \rho_s \ge \alpha_H + \alpha_L. \end{cases}$$

Since $(p_H, p_L; \alpha_H, \alpha_L)$ is an equilibrium, there exists a sequence of (ϵ^k, δ^k)-*ME*, say $(p_H^k, p_L^k; \alpha_H, \alpha_L)$, such that $(p_H, p_L) = \lim_{k\to\infty}(p_H^k, p_L^k)$ where $\lim_{k\to\infty}(\epsilon^k, \delta^k) = (0,0)$. We let $V_H^{sm}(k)$ and $V_L^{sm}(k)$ be the revenue of a high-value and a low-value agent, respectively, according to $(p_H^k, p_L^k; \alpha_H, \alpha_L)$. Then, we have that $V_i^{sm} = \lim_{k\to\infty} V_i^{sm}(k)$ for all $i \in \{D, F\}$.

1 We show that $V_H^{sm} = V_L^{sm} = 0$ by contradiction. Thus, we suppose that either $V_H^{sm} > 0$ or $V_L^{sm} > 0$ is true on the contrary and find a contradiction for any possible price pair (p_H, p_L) satisfying either of these conditions. To this end, we follow a case-by-case analysis:

i. $(\mathbf{R_H - p_H = R_L - p_L})$: Notice that all agents are pooled together in this case and the equilibrium revenue of the high-value agents is $\tau_s p_H^k(\rho_s/(\alpha_H + \alpha_L))$ by Proposition 1. We also should have that $p_H > 0$

to make sure $V_H^{sm} > 0$. Then, consider a small fraction (less than δ^k) of high-value agents deviate and cut their prices by an arbitrarily small $\zeta > 0$. The revenue of deviating agents will be $\tau_s(p_H^k - \zeta)$ for large k by Proposition 1. This deviation improves the revenue of deviating agents because $\rho_s < \alpha_H + \alpha_L$, so that any (p_H, p_L) satisfying $R_H - p_H = R_L - p_L$ cannot emerge as an equilibrium price pair.

ii. $(\mathbf{R_L - p_L > R_H - p_H})$**:** In this case, we have two sub-cases: **a)** $(\boldsymbol{\rho_s \leq \alpha_L})$**:** By Corollary 2, we have that $\lim_{k\to\infty} V_H^{sm}(k) = 0$ since $\rho_s \leq \alpha_L$. We also should have that $p_L > 0$ to make sure $V_L^{sm} > 0$. Then, consider a small fraction of high-value agents deviate and charge p' with $0 < p' < p_L$. The revenue of deviating agents will be p' for large k by Proposition 1. As this deviation improves the revenue of the deviating agents, any (p_H, p_L) in this sub-case cannot emerge as an equilibrium price pair. **b)** $(\boldsymbol{\rho_s > \alpha_L})$**:** By Corollary 2, we have that $\lim\limits_{k\to\infty} V_H^{sm}(k) = \tau_s p_H^k(\rho_s - \alpha_L)/\alpha_H$. Then, as in **Part 1.i**, a small group of high-value agents can improve their revenues by cutting their price. Therefore, any (p_H, p_L) in this sub-case cannot emerge as an equilibrium price pair.

iii. $(\mathbf{R_H - p_H > R_L - p_L})$**:** By Corollary 2, we have that $\lim_{k\to\infty} V_H^{sm}(k) = \tau_s p_H^k(\rho_s/\alpha_H)$ and $\lim_{k\to\infty} V_L^{sm}(k) = 0$. We also should have that $p_H > 0$ to make sure $V_H^{sm} > 0$. Then, as in **Part 1.i**, a small group of high-value agents can improve their revenues by cutting their price.

2. We suppose that either $V_H^{sm} > \tau_s(R_H - R_L)$ or $V_L^{sm} > 0$ on the contrary and follow a case-by-case analysis:

i. $(\mathbf{R_H - p_H \leq R_L - p_L})$**:** The proof is the same as in **Part 1** because we rely on $\rho_s < \alpha_H + \alpha_L$ in **Part 1**, and it is still the case.

ii. $(\mathbf{R_H - p_H > R_L - p_L})$**:** By Corollary 2, we have that $\lim_{k\to\infty} V_H^{sm}(k) = \tau_s p_H^k$ and $\lim_{k\to\infty} V_L^{sm}(k) = 0$ since $\rho_s = \alpha_H$. We should also have that $p_H > R_H - R_L$ to make sure that $V_H^{sm} > \tau_s(R_H - R_L)$. Then, when a small fraction of low-value agents deviate and charge p' with $0 < p' < (R_L - R_H + p_H)$, the revenue of deviating agents will be p' for large k by Proposition 1. As this deviation improves the revenue of the deviating agents, any (p_H, p_L) in this sub-case cannot emerge as an equilibrium price pair.

5. We suppose that either $V_H^{sm} < R_H$ or $V_L^{sm} < R_L$ on the contrary and follow a case-by-case analysis:

i. $(\mathbf{R_H - p_H = R_L - p_L})$**:** Notice that all agents are pooled together in this case and the equilibrium revenue of the high-value agents is $\tau_s p_H^k)$ by Proposition 1 since $\rho_s > \alpha_H + \alpha_L$. Then, consider a small fraction (less than δ^k) of high-value agents deviate and increase their prices by an arbitrarily small $\zeta > 0$. The revenue of deviating agents will be $\tau_s(p_H^k + \zeta)$ for large k by Proposition 1 since $\rho_s > \alpha_H + \alpha_L$. This deviation improves the revenue of the deviating agents, so that any (p_H, p_L) satisfying $R_H - p_H = R_L - p_L$ cannot emerge as an equilibrium price pair.

ii. $(\mathbf{R_L - p_L > R_H - p_H})$**:** By Corollary 2, we have that $\lim_{k\to\infty} V_H^{sm}(k) = \tau_s p_H^k$ and $\lim_{k\to\infty} V_L^{sm}(k) = \tau_s p_L^k$ since $\rho_s > \alpha_H + \alpha_L$. Then, consider a small fraction (less than δ^k) of low-value agents deviate and increase their prices by an arbitrarily small $0 < \zeta < (R_L - p_L) - (R_H - p_H)$. The revenue of deviating agents will be $\tau_s(p_L^k + \zeta)$ for large k by Proposition 1 since $R_L - p_L - \zeta > R_H - p_H$. As this deviation improves the revenue of the deviating agents, any (p_H, p_L) in this sub-case cannot emerge as an equilibrium price pair.

ii. ($\mathbf{R_H - p_H > R_L - p_L}$): Similar to **Part 5.ii**, high-value agents now can improve their revenues by increasing their prices.

The proofs for parts 3, and 4 are similar to the above ones, and thus omitted.

Non-symmetric Market Equilibrium: We do not rule out the existence of a non-symmetric equilibrium outcome, where the same type of agents (high-value or low-value) charge different prices. However, we show that the possibility of non-symmetric equilibrium can be ignored using the results of Proposition 3 as we focus on agent revenues. The following proposition proves that the same type of agents serving the same customer class in any non-symmetric *Market Equilibrium* must earn zero revenue.

PROPOSITION 3. *Let $V^{sm}_{H_n}$ and $V^{sm}_{L_n}$ be the equilibrium revenue of a high-value and a low-value agents in sub-pool n in the simplified marketplace model, respectively. Then, we have that $V^{sm}_{H_n} = 0$ ($V^{sm}_{L_n} = 0$) for all $n \in \{1, \dots, N\}$ if N, the number of different prices announced by the high-value (low-value) agents is two or more.*

The above proposition directly implies that any non-symmetric equilibrium does not affect our results for the revenue of high-value agents in parts 1 and 2 and for the revenue of low-value agents in parts 1-4 because we do not exclude the possibility of zero revenue in these cases. In the remaining cases, we can show that there is not any non-symmetric equilibrium as follows: Suppose there is a non-symmetric equilibrium, where high-value agents charge different prices, when $\rho_s > \alpha_H$ and $R_H > R_L$. By Proposition 3, we should have that all of the high-value agents earn zero in the equilibrium. However, a small group of high-value agents can guarantee a strictly positive revenue by charging $p' = (R_H - R_L)/2$ since $\rho_s > \alpha_H$. Similarly, we can rule out any non-symmetric equilibrium where low-value agents charge different prices if $\rho_s > \alpha_H + \alpha_L$.

S.1.3. Proof of Proposition 3

In this proof, we focus only on the high-value agents. The proof for the low-value agents is the same, and thus omitted.

Let $(\mathbf{r}, \mathbf{y}) \equiv (r_n, y_n)_{n=1}^N$ be a *Market Equilibrium* where y_n is the the fraction of agents offering the net reward r_n and N is the number of different net rewards announced by the agents. Since $(\mathbf{r_i}, \mathbf{y_i})$ is an *Market Equilibrium*, there exists a sequence $(\mathbf{r_i}^k, \mathbf{y_i}^k)$ such that $(\mathbf{r_i}^k, \mathbf{y_i}^k)$ is a (ϵ^k, δ^k)-*ME* where $\lim_{k\to\infty} \epsilon^k = 0$, $\lim_{k\to\infty} \delta^k = 0$, $\mathbf{r_i} = \lim_{k\to\infty} \mathbf{r_i}^k$, and $\mathbf{y_i} = \lim_{k\to\infty} \mathbf{y_i}^k$. Note that we omit the $\mathbf{t}$ vector, which represents the flexible agents decisions about how much capacity they allocate to each class, because we study the simplified marketplace model after the agents make their service decisions.

Let $\mathcal{N}_H = \{n \in \{1, \dots, N\} : \exists \text{ high-skiled agents in sub-pool } n, \}$, $\underline{n} = \min_{n \in \mathcal{N}_H} n$, and $\overline{n} = \min\{n \in \{1, \dots, N\} : r_n < r_{\underline{n}}\}$. Note that we should have $|\mathcal{N}_H| \geq 2$. Otherwise, our claim would hold trivially since we would have that all high-value agents are in the same sub-pool.

We prove our claim by contradiction. Therefore, we suppose $V^{sm}_{H_{\hat{n}}} > 0$ for some $\hat{n} \in \mathcal{N}_H$ and find a contradiction for $\rho_s > \sum_{n=1}^{\underline{n}} y_n$ and $\rho_s \leq \sum_{n=1}^{\underline{n}} y_n$.

When $\rho_s > \sum_{n=1}^{\underline{n}} y_n$, consider a deviation from $(\mathbf{r}^k, \mathbf{y}^k)$ where $\hat{y} < \delta^k$ fraction of high-value agents from sub-pool $\underline{n}$ increase their prices by $\zeta = (r_{\underline{n}} - r_{\overline{n}})/2 > 0$, which must lead to a price in the finite price set for large k as $\lim_{k\to\infty} \epsilon^k = 0$. Then, by Proposition 1, the revenue of deviating agents is $R_H - r^k_{\underline{n}} + \zeta$ for large k since we have that $\rho_s > \sum_{n=1}^{\underline{n}} y_n$. However, this is a contradiction because deviating agents increase their revenues for large k as $\lim_{k\to\infty} \epsilon^k = 0$ because their revenues before deviation can be at most $R_H - r^k_{\underline{n}}$.

When $\rho_s \leq \sum_{n=1}^{\underline{n}} y_n$, we should have that $V^{sm}_{H_n} = 0$ for all $n \in \{\overline{n}, \ldots, N\}$, and thus we should have that $V^{sm}_{H_{\underline{n}}} > 0$, which implies that $R_H - r_{\underline{n}} > 0$ and $\rho_s \geq \sum_{n=1}^{\underline{n}-1} y_n + y^{\Delta}$ for some $y^{\Delta} > 0$ according to Proposition 1. Consider a deviation from $(\mathbf{r}^k, \mathbf{y}^k)$ where $\hat{y} < y^{\Delta}$ fraction of high-value agents from sub-pool n, for some $n \in \mathcal{N}_H$ with $n \geq \overline{n}$, charge a strictly positive price $p' = (R_H - r_{\underline{n}})/2$, which must be in the finite price set for large k as $\lim_{k\to\infty} \epsilon^k = 0$. Then, by Proposition 1, the revenue of deviating agents is p' for large k since we have that $\rho_s \geq \sum_{n=1}^{\underline{n}-1} y_n + \hat{y}$ by the choices of $\hat{y}$ and p'. This is a contradiction because deviating agents increase their revenues (which were zero) by more than ϵ^k for large k.

Once we show contradictions for $\rho_s > \sum_{n=1}^{\underline{n}} y_n$ and $\rho_s \leq \sum_{n=1}^{\underline{n}} y_n$, we should have that $V^{sm}_{H_n} = 0$ for all $n \in \mathcal{N}_H$ when $|\mathcal{N}_H| \geq 2$.

S.1.4. Existence of the equilibrium:

We prove the existence of the equilibrium by constructing one for each of the following three cases:

Case-1 ($\boldsymbol{\rho_s < \alpha_H}$): We show that $(\tilde{p}^k_H, \tilde{p}^k_L; \alpha_H, \alpha_L)$ is a (ϵ^k, δ^k)-*ME* where $\tilde{p}^k_H = \tilde{p}^k_L = 0$, and ϵ^k and δ^k goes to zero as $k \to \infty$. To prove this claim by contradiction, we suppose that $(\tilde{p}^k_H, \tilde{p}^k_L; \alpha_H, \alpha_L)$ is not (ϵ^k, δ^k)-*ME* for $k > K$ for some K. Then, at least one group of agents must have a profitable deviation. Suppose, a $y^k < \delta^k$ fraction of high-value agents improve their revenues by increasing their prices to $p' > 0$. However, the revenue of deviating agents would be zero for large k after such a deviation by Proposition 1 because $\rho_s < \alpha_H$. Similarly, low-value agents cannot improve their revenues by increasing their prices. Thus, $(\tilde{p}^k_H, \tilde{p}^k_L; \alpha_H, \alpha_L)$ is a (ϵ^k, δ^k)-*ME* as $k \to \infty$.

Case-2 ($\boldsymbol{\alpha_H \leq \rho_s < \alpha_H + \alpha_L}$): In this case, we show that $(\tilde{p}^k_H, \tilde{p}^k_L; \alpha_H, \alpha_L)$ is a (ϵ^k, δ^k)-*ME* where $\tilde{p}^k_H = R_H - R_L - \epsilon^k$, $\tilde{p}^k_L = 0$, and ϵ^k and δ^k goes to zero as $k \to \infty$. We first want to note that the revenue of the high-value agents according to $(\tilde{p}^k_H, \tilde{p}^k_L; \alpha_H, \alpha_L)$ is $\tau_s \tilde{p}^k_H$ by Proposition 1 since $\rho_s \geq \alpha_H$.

Similar to the above case, suppose, on the contrary, that $y^k < \delta^k$ fraction of high-value agents improve their revenues by increasing their prices to p'. Notice that p' must be greater than $R_H - R_L$ to be a profitable deviation. However, the revenue of deviating agents would be at most $\tau_s(R_H - R_L)(\rho_s - \alpha_H)/(\alpha_H + \alpha_L)$ for large k after such a deviation by Proposition 1 because $p' > \tilde{p}^k_H$. This deviation does not improve the revenues of the deviating agents because $\rho_s < \alpha_H + \alpha_L$. Similarly, low-value agents cannot improve their revenues by increasing their prices. Thus, $(\tilde{p}^k_H, \tilde{p}^k_L; \alpha_H, \alpha_L)$ is a (ϵ^k, δ^k)-*ME* as $k \to \infty$.

Case-3 ($\boldsymbol{\rho_s \geq \alpha_H + \alpha_L}$): In this case, we show that $(\tilde{p}^k_H, \tilde{p}^k_L; \alpha_H, \alpha_L)$ is a (ϵ^k, δ^k)-*ME* where $\tilde{p}^k_H = R_H - \underline{u}$, $\tilde{p}^k_L = R_L - \underline{u}$, and ϵ^k and δ^k goes to zero as $k \to \infty$. By Proposition 1, the revenues of the high-value and low-value agents are $\tilde{p}^k_H$ and $\tilde{p}^k_L$, respectively. As $\tilde{p}^k_H$ and $\tilde{p}^k_L$ are the highest prices that the agents can charge, there is not any profitable deviations for these prices.

Appendix S.2: Proofs in Section 6

PROPOSITION 4. *Equilibrium revenues of the flexible agents serving the same class must be the same. Furthermore, letting V_{iF}^{me} be equilibrium revenue of a flexible agent serving class $i \in \{A, B\}$, we have that $V_{AF}^{me} = V_{BF}^{me}$.*

S.2.1. Proof of Proposition 2

Using the standard definition of the correlation coefficient, for any given shape parameter η, we have that

$$Corr(S_A, S_B) = \frac{\eta(\eta+3)(3\eta+1)(\eta(4\eta+9)+4)}{4(\eta+1)^2(\eta(\eta+1)(\eta(\eta+3)+4)+1)}$$

As $\eta \to \infty$, we have that $Corr(S_A, S_B) \to 0$ because the denominator is a higher degree polynomial than the numerator is. On the other hand, as $\eta \to 1$, we have that $Corr(S_A, S_B) \to \frac{4\times4\times(13+4)}{4\times4\times(2\times8+1)} = 1$.

Finally, we prove our claim on the independence as follows:

$$\begin{aligned}
\lim_{\eta\to\infty} P(S_A \le s_A, S_B \le s_B) &= \lim_{\eta\to\infty} \int_0^{s_A} ds_A \int_{\min\{s_A^{\eta}, s_B\}}^{\min\{s_A^{1/\eta}, s_B\}} \frac{\eta+1}{\eta-1} ds_B = \int_0^{s_A} ds_A \int_0^{s_B} 1 \\
&= s_A s_B = \lim_{\eta\to\infty} \left(\int_0^{s_A} ds_A \int_{s_A^{\eta}}^{s_A^{1/\eta}} \frac{\eta+1}{\eta-1} ds_B\right)\left(\int_0^{s_B} ds_B \int_{s_B^{\eta}}^{s_B^{1/\eta}} \frac{\eta+1}{\eta-1} ds_A\right) \\
&= \lim_{\eta\to\infty} P(S_A \le s_A) P(S_B \le s_B).
\end{aligned}$$

S.2.2. Proof of Proposition 4

We first focus on our claim about the equilibrium revenues of flexible agents serving the same class. Note that our claim holds trivially if all of flexible agents serving class $i \in \{A, B\}$ charge the same price. Furthermore, Proposition 3 shows that all flexible agents serving class i should earn zero revenue in the equilibrium if they charge two or more prices. Thus, the equilibrium revenues of the flexible agents serving the same class must be the same.

We, next, show that flexible agents earn the same equilibrium revenue even if they serve different classes. We prove our claim by contradiction. Thus, we suppose $V_{AF}^{me} \neq V_{BF}^{me}$. When $V_{AF}^{me} > V_{BF}^{me} \ge 0$, there must be only one sub-pool, say $\tilde{n}$, with flexible agents serving class A by Part 1. We should also have that $\rho_A > \sum_{n=1}^{\tilde{n}} y_{A_n} + y^{\Delta}$ for some $y^{\Delta} > 0$ by Proposition 1. Consider a deviation where a $y^k < \delta^k$ fraction of flexible agents serving class B charge $p' = (V_{AF}^{me} + V_{BF}^{me})/(2\tau_A)$ and exclusively serve class A (Note that p' must be in the finite price set for large k as $\lim_{k\to\infty} \epsilon^k = 0$). By Proposition 1, the revenue of deviating agents should be $\tau_B p'$ for large k since $\rho_A > \rho_{\tilde{n}}^0 + \lim_{k\to\infty} y^k$ as a result of the choices of $\hat{y}$ and p'. This is a contradiction because deviating agents increase their revenues for large k. Similarly, when $V_{BF}^{me} > V_{AF}^{me} \ge 0$, a small group of flexible agents serving class A can improve their revenues. Hence, we should have that $V_{AF}^{me} = V_{BF}^{me}$.

S.2.3. Proof of Theorem 2

Let γ_A be the portion of service capacity that the flexible agents allocate to class A. We prove our claim assuming that $\tau_A \ge \tau_B$. the proof for $\tau_A < \tau_B$ is very similar.

We first note that γ_A must be between $1 - \rho_B$ and ρ_A because otherwise the flexible agents serving one of the classes would earn zero according to Theorem 1. Furthermore, for any $\gamma_A < \rho_A$, agents' revenue from class A will be $\tau_A\mathbf{E}[S_\eta]$ whereas their revenue from $\tau_B\mathbf{E}[S_\eta]$. If $\tau_A=\tau_B$, none of the agents would have profitable deviation from a strategy where $1-\rho_B<\gamma_A<\rho_A$, and thus this would be a *Market Equilibrium.* Then, our claim holds because

$$\mathbf{E}[S_\eta]=\frac{\eta+1}{\eta-1}\int_0^1\int_{s_A^\eta}^{s_A^{1/\eta}} s_B ds_A ds_B=\frac{\eta+1}{2(\eta-1)}\int_0^1(s_A^{2/\eta}-s_A^{2\eta})ds_A=\frac{(\eta+1)^2}{(\eta+2)(2\eta+1)}.$$

On the other hand, If $\tau_A>\tau_B$, $\gamma_A<\rho_A$ could not be sustained as an equilibrium because agents' revenue from class A would be strictly higher. This would create an opportunity for a small group of agents to improve their revenues by serve class A exclusively. Thus, when $\tau_A>\tau_B$, the only equilibrium candidate is $\gamma_A=\rho_A$. Such an equilibrium can be sustained when flexible agents charge $\tau_B\mathbf{E}[S_\eta]/\tau_A$ to serve class A. To be specific, consider a sequence of strategy profiles such that $(r_A^k,y_A^k,t_A^k)=((1-\tau_B/\tau_A)\mathbf{E}[S_\eta],1,\rho_A+2\delta^k)$ and $(r_B^k,y_B^k,t_B^k)=(0,1,1-\rho_A-2\delta^k)$. In other words, flexible agents charge $\tau_B\mathbf{E}[S_\eta]/\tau_A$ and allocate a capacity that is infinitesimally higher than ρ_A to serve class A. They allocate the remaining of their capacity to class B and charge $\mathbf{E}[S_\eta]$.

The above profile is a (δ^k-ϵ^k)-*Market Equilibirum* for large k with $\delta^k\to 0$ as $k\to\infty$ and $\epsilon^k=\tau_B/\tau_A\mathbf{E}[S_\eta]\frac{2\delta^k}{\rho_A+2\delta^k}$: Agents cannot improve their revenue from class B because it is at the highest possible level. Agents are not fully utilized while serving class A but this is not enough to cut their prices because of the choices of ϵ^k. Finally, a $y'\leq\delta^k$ fraction of agents cannot try to increase their prices. After a price increase, their after deviation revenue would be zero according to Proposition 1 since the service capacity available for class A at the lower price will be strictly above ρ_A.

S.2.4. Proof of Theorem 3

We suppose the firm offers Exam A throughout the proof and suppose $\tau_A\geq\tau_B=1$. We first show the $h(\omega,\tau)\equiv\omega^{1/\eta}+\omega^\eta-2\tau\omega=0$, with $\tau\geq 1$, has a unique non-trivial solution $\bar\omega\in(0,1)$. We have that $\frac{\partial^2 h(\omega,tau)}{\partial\omega^2}<0$ for any $\omega<\eta^{-\frac{3\eta}{\eta^2-1}}\in(0,1)$ and $\frac{\partial^2 h(\omega,tau)}{\partial\omega^2}\geq 0$ otherwise. Combining this with the facts that $\frac{\partial h(\omega,tau)}{\partial\omega}\big|_{\omega=0}>0$ and $\frac{\partial h(\omega,tau)}{\partial\omega}\big|_{\omega=1}>0$, we can find two critical levels of ω, ω_1 and ω_2 with $0<\omega_1<\omega_2<1$, such that $\frac{\partial h(\omega,tau)}{\partial\omega}<0$ for any $\omega\in(\omega_1,\omega_2)$ and $\frac{\partial h(\omega,tau)}{\partial\omega}\geq 0$ otherwise. Then, using $h(0,\tau)=0\geq h(1,\tau)$ for any $\tau\geq 1$, we have that $h(\omega,\tau)>0$ for any $\omega\leq\omega_1$, $h(\omega,\tau)<0$ for any $\omega\geq\omega_2$. Furthermore, $h(\omega,\tau)$ is decreasing in ω for any $\omega\in(\omega_1,\omega_2)$. Thus there exists a unique $\bar\omega(\tau)\in(\omega_1,\omega_2)$ with $h(\bar\omega(\tau),\tau)=0$ and $h(\omega,\tau)>0$ for any $\omega<\bar\omega(\tau)$ and $h(\omega,\tau)<0$ for any $\omega>\bar\omega(\tau)$. We illustrate the solution for $h(\omega,1)=0$ in Figure S.1 for various values for η.

We next prove that the interval $[F_\eta^{-1}(1-\rho_A),F_\eta^{-1}(\rho_B)]$ is the *dominating interval.* For all passing levels with $\omega>F_\eta^{-1}(\rho_B)$, dedicated agents earn zero equilibrium revenue according to Theorem 1 because $\alpha_B>\rho_B$. Furthermore, all flexible agents serve class A and charge R_{AF}. Therefore, the total revenue of the marketplace is $\Pi(\omega,0)=\tau_A\int_\omega^1\int_{s_A^\eta}^{s_A^{1/\eta}} s_A f_{A,B} ds_A ds_B$. Notice that $\Pi(\omega,0)$ is decreasing in ω, so that $\Pi(\omega,0)<\Pi(F_\eta^{-1}(\rho_B),0)$ for any $\omega>F_\eta^{-1}(\rho_B)$.

Now, consider the passing levels with $\omega<F_\eta^{-1}(1-\rho_A)$. The total capacity of the flexible agents is above both ρ_A and ρ_B when $\omega<F_\eta^{-1}(1-\rho_A)$. Therefore, their revenue in the equilibrium cannot exceed

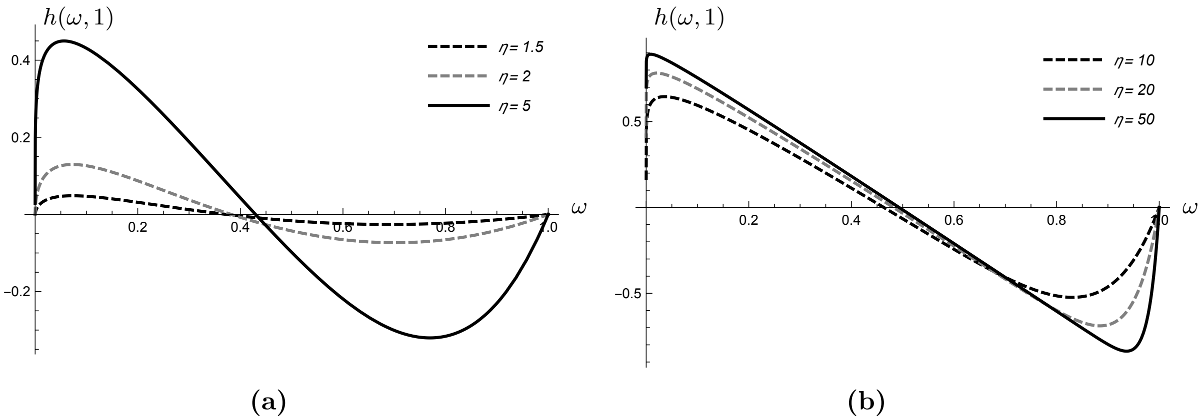

Figure S.1 Illustration of the solution for $h(\omega,1)=0$ when η is a) low and b) high.

$\min\{\tau_A R_{AF}, R_{BF}\}$ because the flexible agents must earn the same equilibrium revenue regardless of the class they serve by Proposition 3. We show that $R_{AF} \geq R_{BF}$: $R_{AF} = R_{BF}$ at $\omega = 0$ and $\omega = 1$. Moreover, $(R_{AF} - R_{BF})\alpha_F$ is increasing in ω for any $\omega < \bar{\omega}(1)$ and decreasing otherwise because its derivative with respect to ω is $\frac{(\eta+1)(\omega^{1/\eta}-\omega^{\eta})}{2(\eta-1)} h(\omega,1)$. Therefore, all flexible agents earn at most R_{BF}. We also have that dedicated agents earn at most R_B. Combining these two, we have that $\Pi(\omega,0) \leq \mathbf{E}[S_\eta] \leq \Pi(F_\eta^{-1}(1-\rho_A),0)$ where the last inequality holds because $R_{AF} \geq R_{BF}$ and all flexible agents can serve class A when $\omega = F_\eta^{-1}(1-\rho_A)$.

After proving $[F_\eta^{-1}(1-\rho_A), F_\eta^{-1}(\rho_B)]$ dominates the rest of the passing levels, we focus on the total revenue in the *dominating interval*. For any $\omega \in [F_\eta^{-1}(1-\rho_A), F_\eta^{-1}(\rho_B)]$, the flexible agents sustain an equilibrium by serving only class A and charging R_{AF} because $\alpha_F \leq \alpha_A$. Moreover, the dedicated agents can charge R_B since $\alpha_B \leq \rho_B$. Therefore, the total revenue or any $\omega \in [F_\eta^{-1}(1-\rho_A), F_\eta^{-1}(\rho_B)]$ is $\Pi(\omega,0) = \int_0^\omega \int_{s_A^\eta}^{s_A^{1/\eta}} s_B f_{A,B} ds_A ds_B + \tau_A \int_\omega^1 \int_{s_A^\eta}^{s_A^{1/\eta}} s_A f_{A,B} ds_A ds_B$. Taking the derivative of the revenue function, we have that $\Pi'(\omega,0) = \frac{(\eta+1)(\omega^{1/\eta}-\omega^{\eta})}{2(\eta-1)} h(\omega,\tau_A)$. Therefore, the optimal passing level is the solution of $h(\omega,\tau_A)=0$ if $\bar{\omega}(\tau_A)$ is inside the dominating interval. Otherwise, one of the end points of the dominating interval is optimal. Finally, the existence of the *Market Equilibrium* is directly due to Theorem 1.

S.2.5. Proof of Theorem 4

We first want to note that $\lim_{\eta\to\infty} \bar{\rho} = \tau_j/(2\tau_i)$ because $\lim_{\eta\to\infty} \bar{\omega} = \tau_j/(2\tau_i)$ and $\lim_{\eta\to\infty} F_\eta(\omega) = \omega$. Therefore, we have that $\lim_{\eta\to\infty} \omega_i^*$ is as described in the theorem. Furthermore, when only Exam i is offered with threshold ω, flexible and dedicated agents cannot earn more than $\tau_i R_{iF}$ and $\tau_j R_j$, respectively. Therefore, as $\eta \to \infty$, the highest possible profit that can be generated in the marketplace cannot exceed $\Pi_i(\omega) \equiv \tau_j\omega/2 + \tau_i \int_\omega^1 s ds$ with $i \neq j \in \{A,B\}$, i.e., $\lim_{\eta\to\infty} \Pi(\omega_A,\omega_B) \leq \overline{\Pi}_i(\omega)$ for any $\omega_i = \omega$ and $\omega_j = 0$. As long as $\omega \in [1-\rho_i, \rho_B]$, the highest profit can actually be sustained as a *Market Equilibrium*, where all flexible agents serve class i. Hence, we also have that $\lim_{\eta\to\infty} \Pi_i^* = \max_{1-\rho_i\leq\omega\leq\rho_j} \overline{\Pi}_i(\omega)$

Using the above observations, we can write the firm's optimal revenue under Exam i as

$$\lim_{\eta\to\infty} \Pi_i^* = \begin{cases} \frac{\tau_j}{2}\left[(1-\rho_j^2)\frac{\tau_i}{\tau_j} + \rho_j\right] & if\ \rho_j \leq \frac{\tau_j}{2\tau_i} \\ \frac{\tau_j}{2}\left[\frac{\tau_i}{\tau_j} + \frac{\tau_j}{4\tau_i}\right] & if\ 1-\rho_i < \frac{\tau_j}{2\tau_i} < \rho_j \\ \frac{\tau_j}{2}\left[\rho_i(2-\rho_i)\frac{\tau_i}{\tau_j} + 1 - \rho_i\right] & if\ \frac{\tau_j}{2\tau_i} \leq 1-\rho_i. \end{cases}$$

Then our claim about the revenue improvement holds true since $\lim_{\eta\to\infty}\Pi^o = \tau_j/2$.

In order to prove that Exam i is the optimal exam, we first note that $\overline{\Pi}_i(\omega)\geq\overline{\Pi}_j(1-\omega)$ for all $0\leq\omega\leq 1$ when $\tau_i\geq\tau_j$ because $\overline{\Pi}_i(\omega)-\overline{\Pi}_j(1-\omega)=\omega(1-\omega)[\tau_i-\tau_j]/2\geq 0$. Then, we prove the optimality of Exam i by considering three cases: (i) $1-\rho_i<\frac{\tau_j}{2\tau_i}<\rho_j$, (ii) $\rho_j\leq\frac{\tau_j}{2\tau_i}$, and (iii) $\frac{\tau_j}{2\tau_i}\leq 1-\rho_i$.

When $1-\rho_i<\frac{\tau_j}{2\tau_i}<\rho_j$, Exam i is optimal because $\lim_{\eta\to\infty}\Pi_i^*=\max_{0\leq\omega\leq 1}\overline{\Pi}_i(\omega)\geq\max_{0\leq\omega\leq 1}\overline{\Pi}_j(\omega)\geq\lim_{\eta\to\infty}\Pi_j^*$. When $\rho_j\leq\frac{\tau_j}{2\tau_i}$, we have that $\lim_{\eta\to\infty}\Pi_i^*=\max_{0\leq\omega\leq\rho_j}\overline{\Pi}_i(\omega)\geq\max_{1-\rho_j\leq\omega\leq 1}\overline{\Pi}_j(\omega)\geq\lim_{\eta\to\infty}\Pi(\omega_A,\omega_B)$ for any $\omega_i=0$ and $1-\rho_j\leq\omega_j\leq 1$. Moreover, when Exam j is offered with threshold $\omega<1-\rho_j$, flexible agents cannot sustain an equilibrium where all of them serve only class j because doing so would leave them with zero revenue according to Theorem 1. Thus, flexible agents' revenues cannot exceed $\tau_i R_i$. The dedicated agents' revenues are also capped by $\tau_i R_i$ because some of them have to serve class i. As a result, we have that $\lim_{\eta\to\infty}\Pi(\omega_A,\omega_B)\leq\tau_i/2=\overline{\Pi}_i(0)$ for any $\omega_i=0$ and $0\leq\omega_j<1-\rho_j$. Combining these two findings, we have that $\lim_{\eta\to\infty}\Pi_i^*\geq\lim_{\eta\to\infty}\Pi(\omega_A,\omega_B)$ for any $\omega_i=0$ and $0\leq\omega_j<1$ when $\rho_j\leq\frac{\tau_j}{2\tau_i}$, which prove the optimality of Exam i for case (ii).

When $\frac{\tau_j}{2\tau_i}\leq 1-\rho_i$, we have that $\lim_{\eta\to\infty}\Pi_i^*=\max_{1-\rho_i\leq\omega\leq 1}\overline{\Pi}_i(\omega)\geq\max_{0\leq\omega\leq\rho_i}\overline{\Pi}_j(\omega)\geq\lim_{\eta\to\infty}\Pi(\omega_A,\omega_B)$ for any $\omega_i=0$ and $0\leq\omega_j\leq\rho_i$. Moreover, when Exam j is offered with threshold $\omega>\rho_i$, dedicated agents earn zero equilibrium revenue according to Theorem 1, so that $\lim_{\eta\to\infty}\Pi(\omega_A,\omega_B)\leq\tau_j\int_\omega^1 s ds=\overline{\Pi}_j(\rho_i)$ for any $\omega_i=0$ and $\rho_i<\omega_j<1$. Combining these two findings, we have that $\lim_{\eta\to\infty}\Pi_i^*\geq\lim_{\eta\to\infty}\Pi(\omega_A,\omega_B)$ for any $\omega_i=0$ and $0\leq\omega_j<1$ when $\frac{\tau_j}{2\tau_i}\leq 1-\rho_i$, which prove the optimality of Exam i for case (iii).

S.2.6. Proof of Theorem 5

We prove our claim assuming $\tau_A\geq\tau_B=1$ for ease of explanation. The proof for $\tau_A<\tau_B$ is almost identical. In our proof, we denote the marginal probability distribution by $f_1(\omega)$ as $\eta\to 1$, which is equal to $4\omega\log(1/\omega)$. Also notice that $\lim_{\eta\to 1}R_{AF}=\lim_{\eta\to 1}R_{BF}$ for any (ω_A,ω_B).

When the firm offers only Exam A, the flexible agents can earn at most $\tau_A R_{AF}$, and the dedicated agents can earn at most R_B simply because these are the highest possible prices that they can charge. Therefore, when the exam threshold is ω_A, we have that

$$\lim_{\eta\to 1}\Pi(\omega_A,0)\leq\lim_{\eta\to 1}\alpha_B R_B+\alpha_F R_{AF}=\int_0^{\omega_A}sf_1(s)ds+\tau_A\int_{\omega_A}^1 sf_1(s)ds,$$

where the the upper-bound is a decreasing function of ω_A. It is also important to note that the flexible agents can sustain an equilibrium where they earn $\tau_A R_{AF}$ only when $\omega_A\geq F_1^{-1}(1-\rho_A)$. Therefore, for any exam threshold $\omega_A\geq F_1^{-1}(1-\rho_A)$, we have that $\lim_{\eta\to 1}\Pi(\omega_A,0)\leq\lim_{\eta\to 1}\Pi(F_1^{-1}(1-\rho_A),0)$. On the other hand, for any exam threshold $\omega_A<F_1^{-1}(1-\rho_A)$, flexible agents cannot sustain an equilibrium where all of them serve only class A because doing so would leave them with zero revenue according to Theorem 1. Thus, the flexible agents can earn at most R_{BF} because some of them have to serve class B. Therefore, we have that $\lim_{\eta\to 1}\Pi(\omega_A,0)\leq\lim_{\eta\to 1}\alpha_B R_B+\alpha_F R_{BF}=\int_0^1 sf_1(s)ds=4/9$. Finally, we have that $\lim_{\eta\to 1}\omega^*=\equiv F_1^{-1}(1-\rho_A)$ because

$$\lim_{\eta\to 1}\Pi(F_1^{-1}(1-\rho_A),0)=\int_0^{\omega_1^*}sf_1(s)ds+\tau_A\int_{\omega_1^*}^1 sf_1(s)ds=\int_0^1 sf_1(s)ds+(\tau_A-1)\int_{\omega_1^*}^1 sf_1(s)ds\geq 4/9,$$

where $\omega_1^* \equiv F_1^{-1}(1-\rho_A)$ and the last inequality holds because $\int_0^1 sf_1(s)ds = 4/9$ and $\tau_A > 1$. The above equation also implies that $\lim_{\eta\to 1} \Delta_A^* = 9/4(\tau_A - 1)\int_{\omega_1^*}^1 sf_1(s)ds$.

When the firm offers only Exam B, flexible agents cannot sustain an equilibrium where all of them serve only class A because doing so would leave them with zero revenue according to Theorem 1. Thus, the flexible agents can earn at most R_{BF}. This also sets a cap for the equilibrium revenues of the dedicated agents serving class A at $\tau_A R_A - (\tau_A - 1)R_{BF}$ because according to Theorem 1, dedicated and flexible serving the same class should leave the same net reward to the customers. Then, we have that

$$\lim_{\eta\to 1}\Pi(0,\omega_B) \le \lim_{\eta\to 1}\alpha_A[\tau_A R_A - (\tau_A - 1)R_{BF}] + R_{BF}\alpha_F = \int_{\omega_B}^1 sf_1(s)ds + \tau_A\int_0^{\omega_B} sf_1(s)ds - (\tau_A - 1)\int_0^{\omega_B} R_{F_1} f_1(s)ds$$
$$\le \int_0^1 sf_1(s)ds + (\tau_A - 1)\int_0^{\omega_B}[s - R_{F_1}]f_1(s)ds \le \int_0^1 sf_1(s)ds = 4/9,$$

where the second inequality holds because $R_{F_1} \equiv \lim_{\eta\to 1} R_{BF} \ge \omega_B$. Combining the above inequality with the fact that $\lim_{\eta\to 1}\Pi(F_1^{-1}(1-\rho_A),0) \ge 4/9$ proves the optimality of offering Exam A.

S.2.7. Proof of Theorem 6

Let $\tilde{\pi}_i(\omega) = \tau_j\omega\big(\int_{\max\{1-\rho_j/\omega,0\}}^1 sds\big) + \tau_i\int_\omega^1 sds$ for $i \ne j \in \{A,B\}$. $\tilde{\pi}_i(\omega)$ is concave in ω. Furthermore, denoting the level of ω_i that maximize $\tilde{\pi}_i(\omega)$ over the range $[1-\rho_i, 1]$ by $\tilde{\omega}$, we have that $\tilde{\omega}_i = \left[\tau_j\rho_j^2/(2\tau_i)\right]^{1/3}$ when $\rho_j < \tau_j/(2\tau_i)$, and at $\tilde{\omega}_i = \min\{1-\rho_i, \tau_j/(2\tau_i)\}$, otherwise. We illustrate the structure of $\tilde{\pi}_i(\omega)$ in Figure S.2.

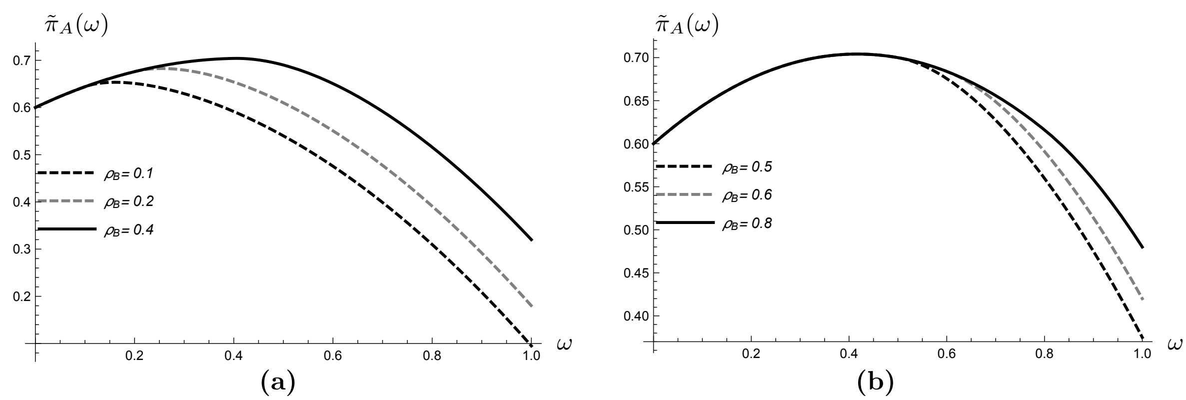

Figure S.2 **Illustration of the function $\tilde{\pi}_A(\omega)$ when $\tau_A = 1.2$, $\tau_B = 1$, and ρ_B is a) less than $\tau_A/(2\tau_B)$ and b) more than $\tau_A/(2\tau_B)$.**

Using these properties of $\tilde{\pi}_i(\omega)$, we show that $\lim_{\eta\to\infty}\Pi(\omega_A,\omega_B)$ is bounded above as the following proposition establishes:

Proposition 5. $\lim_{\eta\to\infty}\Pi(\omega_A,\omega_B) \le \max\{\hat{\pi}_A(\tilde{\omega}_A), \hat{\pi}_B(\tilde{\omega}_B)\}$.

As we use it in our proof, we want to recall that $\overline{\Pi}_i(\omega) \ge \overline{\Pi}_j(1-\omega)$ if $\tau_i \ge \tau_j$ as discussed in the proof of Theorem 4, where $\overline{\Pi}_i(\omega) = \tau_j\omega/2 + \tau_i\int_\omega^1 sds$ for $i \ne j \in \{A,B\}$. Moreover, we have that $\tilde{\pi}_i(\tilde{\omega}_i) \le \max_{1-\rho_i \le \omega \le 1}\overline{\Pi}_i(\omega)$, where the equality holds when $\rho_j \ge \tau_j/(2\tau_i)$.

1. We prove our claim by considering two cases: i) $\tau_A < \tau_B$, and ii) $\tau_A \geq \tau_B$.

i)$\tau_{\mathbf{A}} < \tau_{\mathbf{B}}$: Note that offering Exam B is optimal in the *One Test* case since $\tau_A < \tau_B$. Therefore, we have that $\lim_{\eta\to\infty}\max\{\Pi_A^*,\Pi_B^*\} = \lim_{\eta\to\infty}\Pi_B^*$. Furthermore, we have that $\lim_{\eta\to\infty}\Pi_B^* = \max_{1-\rho_B\leq\omega\leq 1}\overline{\Pi}_B(\omega)$ since $\rho_A \geq 1/2 \geq \tau_A/(2\tau_B)$. Next, we have that $\tilde{\pi}_B(\tilde{\omega}_B) \geq \tilde{\pi}_A(\tilde{\omega}_A)$ because

$$\tilde{\pi}_A(\tilde{\omega}_A) \leq \overline{\Pi}_A(\tau_B/(2\tau_A)) \leq \overline{\Pi}_B(1-\tau_B/(2\tau_A)) \leq \max_{1-\rho_B\leq\omega\leq 1}\overline{\Pi}_B(\omega) = \tilde{\pi}_B(\tilde{\omega}_B).$$

The second inequality above holds because $\tau_A < \tau_B$ implies that $\overline{\Pi}_B(\omega) \geq \overline{\Pi}_A(1-\omega)$. The third one holds because we have that $1-\rho_B \leq 1-\tau_B/(2\tau_A)$, which is a direct implication of $\rho_B \geq \tau_B/(2\tau_A)$.

Combining these observations with Proposition 5, we have that $\lim_{\eta\to\infty}\frac{\Pi(\omega_A,\omega_B)}{\max\{\Pi_A^*,\Pi_B^*\}} \leq 1$ when $\rho_B \geq \tau_B/(2\tau_A)$, which implies that $\lim_{\eta\to\infty}\Delta^{**} = 0$.

ii)$\tau_{\mathbf{A}} \geq \tau_{\mathbf{B}}$: Similar to the previous case, Exam A is the optimal exam, and thus we have that $\lim_{\eta\to\infty}\max\{\Pi_A^*,\Pi_B^*\} = \lim_{\eta\to\infty}\Pi_A^*$. Furthermore, we have that $\lim_{\eta\to\infty}\Pi_A^* = \max_{1-\rho_A\leq\omega\leq 1}\overline{\Pi}_A(\omega)$ since $\rho_B \geq \tau_B/(2\tau_A)$. Next, we show that $\tilde{\pi}_A(\tilde{\omega}_A) \geq \tilde{\pi}_B(\tilde{\omega}_B)$. First, when $1-\rho_A \leq \tau_B/(2\tau_A)$, we have that

$$\tilde{\pi}_B(\tilde{\omega}_B) \leq \overline{\Pi}_B(\tau_A/(2\tau_B)) \leq \overline{\Pi}_A(1-\tau_A/(2\tau_B)) \leq \overline{\Pi}_A(\tau_B/(2\tau_A)) \leq \max_{1-\rho_A\leq\omega\leq 1}\overline{\Pi}_A(\omega) = \tilde{\pi}_A(\tilde{\omega}_A).$$

When $1-\rho_A > \tau_B/(2\tau_A)$, which also implies that $\rho_A < \min\{1,\tau_A/(2\tau_B)\}$, we have that

$$\begin{aligned}\tilde{\pi}_B(\tilde{\omega}_B) - \tilde{\pi}_A(\tilde{\omega}_A) &\leq \max_{\substack{(\rho_A,\tau_A,\tau_B)\\ 1/2\leq\rho_A\leq\tilde{\omega}_B\leq 1}} \tilde{\pi}_B(\tilde{\omega}_B) - \tilde{\pi}_A(\tilde{\omega}_A) \leq \max_{\substack{(\rho_A,\tau_A,\tau_B)\\ 1/2\leq\rho_A=\tilde{\omega}_B\leq 1}} \tilde{\pi}_B(\tilde{\omega}_B) - \tilde{\pi}_A(\tilde{\omega}_A)\\ &= \max_{1/2\leq\rho_A\leq 1}\rho_A(1-\rho_A)(1-2\rho_A)\tau_B/2 \leq 0,\end{aligned}$$

where the second inequality holds because $\tilde{\pi}_B(\tilde{\omega}_B) - \tilde{\pi}_A(\tilde{\omega}_A)$ is decreasing in τ_A, which implies that the maximum must be achieved when $\tilde{\omega}_B = \rho_A$, and the last one holds because $1/2 \leq \rho_A \leq 1$.

Finally, we want to note that $\lim_{\eta\to\infty}\Pi_A^* = \max_{1-\rho_A\leq\omega\leq 1}\overline{\Pi}_A(\omega) = \tilde{\pi}_A(\tilde{\omega}_A)$.

Combining these observations with Proposition 5, we have that $\lim_{\eta\to\infty}\Delta^{**} = 0$ as in the case of $\tau_A < \tau_B$.

2. We prove our claim by considering two cases: i) $\tau_A < \tau_B$, and ii) $\tau_A \geq \tau_B$.

i) $\tau_{\mathbf{A}} < \tau_{\mathbf{B}}$: As in part 1, Exam B is the optimal exam, and thus we have that $\lim_{\eta\to\infty}\max\{\Pi_A^*,\Pi_B^*\} = \lim_{\eta\to\infty}\Pi_B^*$. Furthermore, we have that $\lim_{\eta\to\infty}\Pi_B^* = \max_{1-\rho_B\leq\omega\leq 1}\overline{\Pi}_B(\omega)$ since $\rho_A \geq 1/2 > \tau_A/(2\tau_B)$. We also want to note that if $\tilde{\pi}_B(\tilde{\omega}_B) \geq \tilde{\pi}_A(\tilde{\omega}_A)$, we would have that $\lim_{\eta\to\infty}\Delta^{**} = 0$ because $\tilde{\pi}_B(\tilde{\omega}_B) = \max_{1-\rho_B\leq\omega\leq 1}\overline{\Pi}_B(\omega)$ due to the fact that $\rho_A > \tau_A/(2\tau_B)$. Thus, we must have that $\tilde{\pi}_B(\tilde{\omega}_B) < \tilde{\pi}_A(\tilde{\omega}_A)$ in order to have any benefits from the second exam. Moreover, we would have that $\lim_{\eta\to\infty}\Delta^{**} = 0$ if $1-\rho_B \leq \tau_A/(2\tau_B)$ because $1-\rho_B \leq \tau_A/(2\tau_B)$ implies that $\lim_{\eta\to\infty}\Pi_B^* = \max_{0\leq\omega\leq 1}\overline{\Pi}_B(\omega) \geq \max_{0\leq\omega\leq 1}\overline{\Pi}_A(\omega) \geq \tilde{\pi}_A(\tilde{\omega}_A)$.

Using these observations, we have that

$$\lim_{\eta\to\infty}\Delta^{**} \leq \max_{\substack{(\rho_B,\tau_A,\tau_B)\\ 0\leq\rho_B\leq\tilde{\omega}_A\leq 1\\ \tilde{\pi}_A(\tilde{\omega}_A)\geq\overline{\Pi}_B(1-\rho_B)}} \frac{\tilde{\pi}_A(\tilde{\omega}_A)}{\overline{\Pi}_B(1-\rho_B)} - 1 \leq \max_{\substack{(\rho_B,\tau_B)\\ 0\leq\rho_B\leq\tilde{\omega}_A\\ \rho_B\leq 1/2,\ \tau_A=\tau_B}} \frac{\tilde{\pi}_A(\tilde{\omega}_A)}{\overline{\Pi}_B(1-\rho_B)} - 1 = \max_{0\leq\rho_B\leq 1/2}\frac{\rho_B(2-3\sqrt[3]{2\rho_B}+2\rho_B)}{2+2(1-\rho_B)\rho_B} \leq 0.021.$$

The second inequality above holds because $\frac{\tilde{\pi}_A(\tilde{\omega}_A)}{\overline{\Pi}_B(1-\rho_B)}$ is decreasing in τ_B, which implies that the maximum must be achieved when $\tau_B = \min\{\rho_B\tau_A/2, \tau_A\}$. We also show that we should have that $\rho_B \le 1/2$ in order to have $\tilde{\pi}_A(\tilde{\omega}_A) \ge \overline{\Pi}_B(1-\rho_B)$, and thus we should have that $\tau_B = \tau_A$.

ii) $\tau_{\mathbf{A}} \ge \tau_{\mathbf{B}}$: As in part 1, Exam A is the optimal exam, and thus we have that $\lim_{\eta\to\infty}\max\{\Pi_A^*, \Pi_B^*\} = \lim_{\eta\to\infty}\Pi_A^*$. Furthermore, we have that $\lim_{\eta\to\infty}\Pi_B^* = \overline{\Pi}_A(\rho_B)$ since $\rho_B \le \tau_B/(2\tau_A)$. We also want to note that $\tilde{\pi}_A(\tilde{\omega}_A) \ge \tilde{\pi}_B(\tilde{\omega}_B)$ because

$$\tilde{\pi}_A(\tilde{\omega}_A) \ge \max_{0\le\omega\le\rho_B}\overline{\Pi}_A(\omega) \ge \max_{1-\rho_B\le\omega\le1}\overline{\Pi}_B(\omega) = \tilde{\pi}_B(\tilde{\omega}_B).$$

The first inequality above holds since $\overline{\Pi}_A(\omega)$ is increasing in ω when $\rho_B \le \tau_B/(2\tau_A)$. The second one holds because $\tau_A \ge \tau_B$ implies that $\overline{\Pi}_A(\omega) \ge \overline{\Pi}_B(1-\omega)$.

Using these observations and the fact that $\rho_B \le \tau_B/(2\tau_A) \le 1/2$, we have that

$$\lim_{\eta\to\infty}\Delta^{**} \le \max_{\substack{(\rho_B,\tau_A,\tau_B)\\ 0\le\rho_B\le\tilde{\omega}_A\le1\\ \rho_B\le1/2}} \frac{\tilde{\pi}_A(\tilde{\omega}_A)}{\overline{\Pi}_A(\rho_B)} - 1 \le \max_{\substack{(\rho_B,\tau_B)\\ 0\le\rho_B\le\tilde{\omega}_A\\ \rho_B\le1/2,\ \tau_A=\tau_B}} \frac{\tilde{\pi}_A(\tilde{\omega}_A)}{\overline{\Pi}_A(\rho_B)} - 1 = \max_{0\le\rho_B\le1/2} \frac{\rho_B(2-3\sqrt[3]{2\rho_B}+2\rho_B)}{2+2(1-\rho_B)\rho_B} \le 0.021$$

The second inequality above holds because $\frac{\tilde{\pi}_A(\tilde{\omega}_A)}{\overline{\Pi}_A(\rho_B)}$ is decreasing in τ_A, which implies that the maximum must be achieved when $\tau_A = \tau_B$.

3. We prove our claim assuming $\tau_A \ge \tau_B = 1$ for ease of explanation. The proof for $\tau_A < \tau_B$ is almost identical. In our proof, we denote the marginal probability distribution by $f_1(\omega)$ as $\eta \to 1$. Also notice that $\lim_{\eta\to1} R_{AF} = \lim_{\eta\to1} R_{BF}$ for any (ω_A, ω_B).

As the first step of our proof, we show that $\lim_{\eta\to1}\Pi(\omega_A,\omega_B) \le 4/9$ for any (ω_A,ω_B) with $\omega_B > \omega_A$. When $\omega_B > \omega_A$, the flexible agents can earn at most R_{BF} since $\tau_A \ge \tau_B$ and R_{AF} and R_{BF} converges to each other as $\eta \to 1$. Furthermore, we have dedicated agents serving class A and their equilibrium revenue cannot exceed $\tau_A R_A - (\tau_A - 1)R_{BF}$. Therefore, we have that $\lim_{\eta\to1}\Pi(\omega_A,\omega_B) \le \lim_{\eta\to1}\alpha_A[\tau_A R_A - (\tau_A-1)R_{BF}] + R_{BF}\alpha_F = \int_{\omega_A}^1 sf_1(s)ds + (\tau_A-1)\int_{\omega_A}^{\omega_B}[s-R_{BF}]f_1(s)ds \le \int_{\omega_A}^1 sf_1(s)ds \le 4/9$, where the second inequality holds because $R_{BF} \ge \omega_B$.

For any (ω_A,ω_B) with $\omega_A > \omega_B$ and $\omega_A \ge F_1^{-1}(1-\rho_A)$, the flexible agents can earn at most $\tau_A R_{AF}$, and the dedicated agents can earn at most R_B. Therefore, we have that

$$\begin{aligned}\lim_{\eta\to1}\Pi(\omega_A,\omega_B) &\le \lim_{\eta\to1}\alpha_B R_B + \alpha_F R_{AF} = \int_{\omega_B}^{\omega_A} sf_1(s)ds + \tau_A\int_{\omega_A}^1 sf_1(s)ds \le \int_0^{F_1^{-1}(1-\rho_A)} sf_1(s)ds + \tau_A\int_{F_1^{-1}(1-\rho_A)}^1 sf_1(s)ds\\ &= \lim_{\eta\to1}\Pi(F_1^{-1}(1-\rho_A),0),\end{aligned}$$

where the second inequality holds because the left-hand-side decreases in ω_B and ω_A. On the other hand, when $\omega_A < F_1^{-1}(1-\rho_A)$, the flexible agents can earn at most R_{BF} because some of them have to serve class B. Therefore, we have that $\lim_{\eta\to1}\Pi(\omega_A,\omega_B) \le \lim_{\eta\to1}\alpha_B R_B + \alpha_F R_{BF} = \int_{\omega_B}^1 sf_1(s)ds \le 4/9$.

Combining the above observations, we have that $\lim_{\eta\to1}\Pi(\omega_A,\omega_B) \le \max\{4/9, \lim_{\eta\to1}\Pi(F_1^{-1}(1-\rho_A),0)\}$ for any (ω_A,ω_B). As we discuss in the proof of Theorem 5, $\lim_{\eta\to1}\Pi(F_1^{-1}(1-\rho_A),0) \ge 4/9$, and the firm can achieve $\lim_{\eta\to1}\Pi(F_1^{-1}(1-\rho_A),0)$ using only one test. Hence, the second exam does not bring any benefits, i.e., $\lim_{\eta\to1}\Delta^{**} = 0$.

S.2.8. Proof of Proposition 5

We will follow a case-by-case analysis based on the regions described in Figure S.3 to prove that

$$\lim_{\eta\to\infty}\Pi(\omega_A,\omega_B)\leq\max\{\hat{\pi}_A(\tilde{\omega}_A),\hat{\pi}_B(\tilde{\omega}_B)\},$$

where $\tilde{\pi}_i(\omega)=\tau_j\omega\big(\int_{\max\{1-\rho_j/\omega,0\}}^1 sds\big)+\tau_i\int_\omega^1 sds$ for $i\neq j\in\{A,B\}$. Furthermore, we let $\tilde{\omega}\equiv\arg\max_{1-\rho_i\leq\omega\leq 1}\tilde{\pi}_i(\omega)$. We focus on the case of $\tau_A\geq\tau_B$ because the proof is almost identical for $\tau_A\leq\tau_B$.

For notational convenience, we use the upper-script ˜ to denote the limit of the revenue function, expected reward functions, and fraction of agents as $\eta\to\infty$. We also note that the customers expect the same reward from flexible and dedicated agents at the the limit. Therefore, we denote the class $i\in\{A,B\}$ customers' expected reward at the limit by $\tilde{R}_i$.

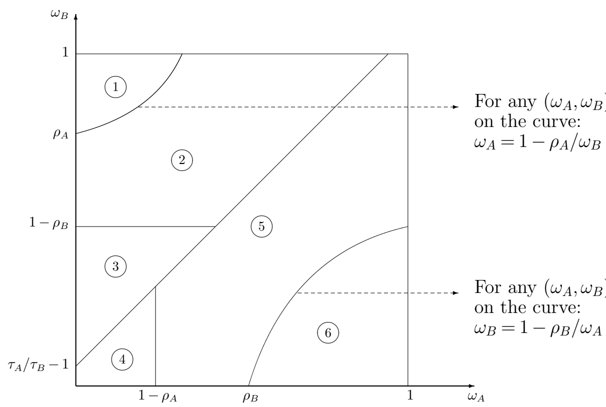

Figure S.3 **Different regions that a given passing levels (ω_A,ω_B) falls as $\eta\to\infty$ when $\tau_A\geq\tau_B$.**

Region-1: For any (ω_A,ω_B) in this region, we have that $\tilde{\alpha}_A>\rho_A$. Therefore, the equilibrium revenue of dedicated agents serving class A should be zero according to Theorem 1. Furthermore, agents serving class B cannot earn more than $\tau_B\tilde{R}_B$ Then, we have that $\tilde{\Pi}(\omega_A,\omega_B)\leq\tau_B\tilde{R}_B[\tilde{\alpha}_B+\tilde{\alpha}_F]=\tau_B\int_{\omega_B}^1 sds\leq\tilde{\pi}_B(\omega_B)\leq\max_{\rho_A\leq\omega\leq1}\tilde{\pi}_B(\omega)\leq\hat{\pi}(\tilde{\omega}_B)$.

Region-2: For any (ω_A,ω_B) in this region, we have that $\tau_A\tilde{R}_A\leq\tau_B\tilde{R}_B$ since $\tilde{R}_i=(1+\omega_i)/2$ and $\omega_A+\tau_A/\tau_B-1\leq\omega_B$. Therefore, the equilibrium revenue of the flexible agents cannot exceed $\tau_B\tilde{R}_B$. Furthermore, the equilibrium revenue of dedicated agents serving class $i\in\{A,B\}$ cannot exceed $\tilde{R}_i$. Thus, for any (ω_A,ω_B) in this region, we have that $\tilde{\Pi}(\omega_A,\omega_B)\leq\tau_B\tilde{R}_B[\tilde{\alpha}_B+\tilde{\alpha}_F]+\tau_A\tilde{R}_A\tilde{\alpha}_A=\tau_B\int_{\omega_B}^1 sds+\tau_A\omega_B\int_{\omega_A}^1 sds\leq\tilde{\pi}_B(\omega_B)$, where the last inequality holds because left-hand-side is decreasing in ω_A. This implies that $\tilde{\Pi}(\omega_A,\omega_B)\leq\max_{1-\rho_B\leq\omega\leq1}\tilde{\pi}_B(\omega_B)\leq\hat{\pi}(\hat{\omega}_B)$ in Region 2.

The line with $\omega_\mathbf{B}=\omega_\mathbf{A}+\tau_\mathbf{A}/\tau_\mathbf{B}-1$: For any (ω_A,ω_B) in this line, we have that $\tau_A\tilde{R}_A=\tau_B\tilde{R}_B$. Therefore, any agent (flexible or dedicated) serving class $i\in\{A,B\}$ cannot earn more than τ_iR_i. Thus, for any (ω_A,ω_B) in this line, we have that $\tilde{\Pi}(\omega_A,\omega_B)\leq\tau_B\tilde{R}_B[\tilde{\alpha}_B+\tilde{\alpha}_F]+\tau_A\tilde{R}_A\tilde{\alpha}_A=\tau_B\int_{\omega_B}^1 sds+\tau_A\omega_B\int_{\omega_A}^1 sds\leq\tilde{\pi}_B(\omega_B)$. Similarly, we have that $\tilde{\Pi}(\omega_A,\omega_B)\leq\tau_A\tilde{R}_A[\tilde{\alpha}_A+\tilde{\alpha}_F]+\tau_B\tilde{R}_B\tilde{\alpha}_B=\tau_A\int_{\omega_A}^1 sds+\tau_B\omega_A\int_{\omega_B}^1 sds\leq\tilde{\pi}_A(\omega_A)$. In other words, we have that $\tilde{\Pi}(\omega_A,\omega_B)\leq\min\{\tilde{\pi}_A(\omega_A),\tilde{\pi}_B(\omega_B)\}$.

We also note that we should have either $1-\rho_A \leq \tau_B/(2\tau_A)$ or $1-\rho_B \leq \tau_A/(2\tau_B)$ because otherwise we would have that $2-\rho_A-\rho_B > \tau_A/(2\tau_B)+\tau_B/(2\tau_A) \geq 1$, which contradictions with the fact that $\rho_A+\rho_B \geq 1$.

If $1-\rho_A \leq \tau_B/(2\tau_A)$, we have that $\tilde{\pi}_A(\tilde{\omega}) = \max_{0\leq\omega\leq 1} \tilde{\pi}_A(\omega)$ because $\tilde{\pi}_A(\omega)$ is increasing in ω for any $\omega < 1-\rho_A$. Combining this with the above upper-bound on $\tilde{\Pi}(\omega_A,\omega_B)$, we have that $\tilde{\Pi}(\omega_A,\omega_B) \leq \tilde{\pi}_A(\tilde{\omega})$. Similarly, if $1-\rho_B \leq \tau_A/(2\tau_B)$, we have that $\tilde{\Pi}(\omega_A,\omega_B) \leq \tilde{\pi}_B(\tilde{\omega}_B)$.

Region-3: For any (ω_A,ω_B) in this region, we have that $\tilde{\alpha}_B+\tilde{\alpha}_F > \rho_B$ and $\tau_B R_B \geq \tau_A R_A$. Therefore, the flexible agents have to serve both classes, which means that the equilibrium revenue of the flexible agents cannot exceed $\tau_A R_A$. This implies that the revenue of dedicated agents serving class B cannot exceed $\tau_A R_A$ because according to Theorem 1, dedicated and flexible serving the same class should leave the same net reward to the customers. $\Pi(\omega_A,\omega_B) \leq \tau_A R_A[\alpha_A+\alpha_B+\alpha_F] = \tau_A \frac{1-\omega_A\omega_B}{1-\omega_A}\int_{\omega_A}^1 s ds \leq \Pi(\omega_A,\omega_{B_L})$ with $\omega_{B_L} = \omega_A+\tau_A/\tau_B-1$, where the last inequality holds because the left-hand-side is decreasing in ω_B. Then, our claim holds since we already show that $\Pi(\omega_A,\omega_{B_L}) \leq \max\{\hat{\pi}_A(\tilde{\omega}_A),\hat{\pi}_B(\tilde{\omega}_B)\}$ above.

Regions 4, 5, and 6: The proofs for regions 4, 5, and 6 are almost identical to the proofs for regions 3, 2, and 1, respectively, and thus omitted.

S.2.9. Proof of Corollary 1

As we have that $\lim_{\eta\to 1}\Delta_A^* = \lim_{\eta\to 1}\Delta_B^* = \lim_{\eta\to 1}\Delta^{**} = 0$, for any $C>0$ there exists a $\underline{\eta}$ such that both $\Delta^* < C/\Pi^o$ and $\Delta^{**} < C/\max\{\Pi_A^*,\Pi_B^*\}$ holds true for any $\eta < \underline{\eta}$. This implies that $\Pi^o > \Pi_i^* - C$ for all $i \in \{A,B\}$ and $\Pi^o > \max\{\Pi_A^*,\Pi_B^*\} - C > \Pi^{**} - 2C$. Hence, offering zero test is optimal for any $\eta < \underline{\eta}$.

Regarding the second claim, as long as C is between the bounds stated in the corollary, there exists a $\overline{\eta}$ such that both $\Delta^{**} < C/\Pi^{**} \leq C/\max\{\Pi_A^*,\Pi_B^*\}$ and $\Delta^* > C/\Pi^o$ hold true for any $\eta > \overline{\eta}$. This implies that $\max\{\Pi_A^*,\Pi_B^*\} > \Pi^{**} - C$ and $\max\{\Pi_A^*,\Pi_B^*\} - C > \Pi^o$ for any $\eta > \overline{\eta}$. Hence, offering only one test is optimal for any $\eta > \overline{\eta}$.